There's Something About Fibre Bundles

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1 Introduction

The theory of fibre bundles is beautiful mathematics but also plays a prominent role in contemporary theoretical physics. Most of the fundamental physical theories, including the subtheories of the standard model and general relativity, are gauge field theories and the theory of fibre bundles provide a natural mathematical framework for these theories. The fundamental laws appear as differential forms defined on a vector bundle and the solutions to the equations are sections of the same vector bundle. There are a variety of interesting philosophical questions associated with gauge field theories, such as: Why do so many physical theories have gauge freedom? What is the role of symmetry in physics? What is the ontological status of gauge fields? The simplicity, generality and unifying character of the fibre bundle formalism should facilitate inquiry into these questions.

This paper is not concerned with examining these very difficult questions, but rather to examine some related philosophical issues from the point of view of the fibre bundle formalism and its relevance to the philosophy of physics and mathematics. Following an exposition of the formalism, I will examine the role that fibre bundles can play in framing questions regarding the ontology of gauge potentials and how ontological issues together with them lead to a consideration of general relativity as a gauge theory, as well as their general unifying character in both physics and mathematics.

2 The Fibre Bundle Formalism for Gauge Field Theory

2.1 Lie Groups and Lie Algebras

Since gauge theories all involve invariance under some kind of symmetry operation, gauge theory naturally involves group theory in a fundamental way. The groups that are important for gauge theories also have the structure of a manifold and so the notions of a Lie group and a Lie algebra are fundamentally important in gauge field theories. This section is devoted to a summary of some of the pertinent aspects of Lie group theory.

A *real Lie group* G_L is a group that is also a manifold such that the mapping from any element to its inverse and the mapping from any two elements to their product are smooth maps. This last condition is equivalent to the condition that the mapping $(g_1, g_2) \rightsquigarrow g_1 g_2^{-1}$ is C^∞. As a result of the smoothness of the group opertations, two classes of maps, the *right* and *left translations* of G_L, become important.[1] The most common examples of Lie groups, and ones that are particularly important in theoretical physics, are subgroups of the general linear groups $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ of inverible $n \times n$ matricies (with elements in $\mathbb{R}$ and $\mathbb{C}$ respectively). The general linear groups themselves and many of their subgroups, such as the special linear group $SL(n)$ (with elements in either $\mathbb{R}$ or $\mathbb{C}$), the orthogonal group $O(n)$, the special orthogonal group $SO(n)$, the unitary group $U(n)$, and the special unitary group $SU(n)$, are Lie groups.

Just as in the theory of groups, the concept of a homomorphism is important in Lie group theory. A homomorphism of Lie groups is a homomorphism of groups $\rho\colon\ G \longrightarrow G'$ that is also a smooth map between the underlying manifolds of the two Lie groups. Using this definition it is possible to 'represent' Lie group elements by linear transformations on some vector space V. Such a mapping is called a *Lie group representation*, which is a group representation where the homomorhpism ρ is a homomorphism of Lie groups, viz. $\rho\colon\ G \longrightarrow GL(V)$, where $GL(V)$ denotes the general linear group of V, is a Lie group homomorphism. It turns out that the map $G \times V \longrightarrow V$ defined by $(g, v) \rightsquigarrow \rho(g)v$, where $g \in G$ and $v \in V$, is a *group action* because the map satisfies the requisite axiom $(gg', v) = (g, (g', v))$ for any $g, g' \in G$ and $v \in V$.[2]

Lie groups are complicated objects and can be rather difficult to study directly, consequently the Lie Algebra corresponding to a Lie group becomes important because it is closely related to the Lie group but easier to study. Much of this relative ease derives from the fact that the Lie Algebra has an underlying vector space structure. If G is a Lie group, then the *Lie algebra* of G, denoted $\mathfrak{g}$, is the tangent space of the identity element of G, T_eG. The Lie algebra of a Lie group G encodes most of the structure of the entire Lie group G, including the group structure and many of the topological properties of G.

Since a Lie group G is a manifold, we can consider vector fields on G. The vector fields of interest in connection to Lie algebras are left-invariant and right-invariant vector fields; we will focus on the former of the two. A vector field X on a Lie group G is *left-invariant* if it is l_g-related to itself for all $g \in G$, i.e.

$$l_{g_*}X = X \text{ for all } g \in G, \tag{1}$$

[1]See Appendix B for definitions.

[2]This simply follows from the fact that ρ is a homomorphism and associativity.

where l_{g*} is the pushing forward of l_g, the left translation map defined by (17) in Appendix B. The set of all left-invariant vector fields on a Lie group G is denoted $L(G)$ and is a real vector space. It is easy to show that for any two left-invariant vector fields X_1 and X_2, their *commutator* or *Lie bracket* $[X_1, X_2] \equiv X_1X_2 - X_2X_1$ is also a left-invariant vector field. This fact turns $L(G)$ into an algebra, which, astonishingly, is isomorphic to $\mathfrak{g}$. Accordingly, we may also consider $L(G)$ to be the Lie algebra of G. This isomorphism allows the definition of a Lie bracket on the Lie algebra. If we let the left-invariant vector field L^A be the image of $A \in T_eG$ under the isomorphism and if we let $A, A' \in T_eG$ then the Lie bracket $[A, A'] \in T_eG$ is defined to be the unique element in T_eG such that

$$L^{[A,A']} = [L^A, L^{A'}], \tag{2}$$

which is satisifed by

$$[A, A'] \equiv [L^A, L^{A'}]_e. \tag{3}$$

This turns T_eG explicitly into an algebra. With the Lie Bracket defined it is possible to define a homomorphism of Lie algebras as a linear map ρ: $\mathfrak{g} \longrightarrow \mathfrak{h}$ such that $\rho([A, A']) = [\rho(A), \rho(A')]$ for all $A, A' \in \mathfrak{g}$.

An important property of Lie Algebras is that if $\{E_1, E_2, \ldots, E_n\}$, where $n = \dim(G)$, is a basis set for $L(G) \cong T_eG$, then the commutator of any of these fields must be a linear combination of them. This enables us to write

$$[E_\alpha, E_\beta] = C_{\alpha\beta}{}^\gamma E_\gamma, \tag{4}$$ [3]

for some $C_{\alpha\beta}{}^\gamma \in \mathbb{R}$. The numbers $C_{\alpha\beta}{}^\gamma$ are called the *structure constants* of the Lie group or Lie algebra because they characterize the structure of the Lie group.

2.2 Fibre Bundles

A natural mathematical framework for gauge field theories is that of the fibre bundle formalism. It effects a clear separation of the kinematics, supplied by the structure of the base manifold, which usually represents a physical space or spacetime,[4] and the dynamics, supplied by the specification of a Lagrangian. In the case of Yang-Mills theories, for instance, the group of symmetries of the Lagrangian, the internal symmetry, is made local by constructing a fibre bundle with the fibre being the symmetry group. This, then, enables sections, a connection and curvature to be defined on the bundle which represent physical fields on the base manifold, or physical spacetime. Although

[3]Where γ ranges from 1 to n and we are using the Einstein summation convention.

[4]The 'physical space or spacetime' must come with a metric, even though this additional structure is not necessesary for the definition of a bundle, because this metric is necessary for a meaningful kinematics.

general relativity cannot be formulated as a Yang-Mills theory, it can also be formulated using fibre bundles, making the theory of fibre bundles a uniform framework to treat all the fundamental fields in modern physics.

A *bundle* is defined to be a structure (E, π, M) consisting of a smooth manifold E, a smooth manifold M and an onto smooth map $\pi \colon E \longrightarrow M$.[5] The manifold E is called the *total space* or *bundle space*, the manifold M is called the *base space*, and π is called the *projection map*. It is common to refer to the bundle as E, when it is clear what the base space and projection map are. For each point $p \in M$, the inverse image of p under π

$$E_p = \{q \in E : \pi(q) = p\}$$

is called the *fibre over p*. Thus, we have that

$$E = \bigcup_{p \in M} E_p.$$

If for each $p \in M$ E_p is homeomorphic to a common space F, then F is known as the *fibre* of the bundle and the bundle is called a *fibre bundle*. An important example of a fibre bundle is the *tangent bundle* TM formed from the set of all tangent spaces T_pM of M:

$$TM = \bigcup_{p \in M} T_pM,$$

where the projection map $\pi \colon TM \longrightarrow M$ is the map from each tangent vector $v_p \in T_pM$ to the point $p \in M$ and the fibre of the bundle is $\mathbb{R}^n$.

The equivalent of a homomorphism in the case of bundles is a bundle *morphism*, which, given two bundles (E, π, M) and (E', π', M'), is a map $\psi \colon E \longrightarrow E'$ together with a map $\phi \colon M \longrightarrow M'$ such that ψ maps each fibre E_p into the fibre $E'_{\phi(p)}$. It turns out that the map ψ completely determines the morphism. A bundle morphism is an isomorphism if both ϕ and ψ are diffeomorphisms. One can construct a smaller version of a bundle, called the *resriction* of a bundle (E, π, M) to a submanifold $S \subseteq M$, from total space $E|_S = \{q \in E | \pi(q) \in S\}$, base space S and projection map $\pi|_S$.

In general, fibre bundles are extrordinarily complex objects. There are, however, simple cases called *trivial bundles*. Such bundles (E, π, M) have the property that they are isomorphic to a *product bundle* $(M \times F, pr, M)$ for some space F. A more complex,

[5]The resriction that E, M and π be smooth technically makes the bundle in question a C^∞*-bundle*. For E and M topological manifolds and π continuous, (E, π, M) would be a *bundle*. But since we are will always have the former restrictions the will be no confusion caused by calling C^∞-bundles 'bundles.'

but more interesting case is that of a locally trivial bundle. A bundle is called *locally trivial* with standard fibre F if for each point $p \in M$, there is a neighborhood U of p and a bundle isomorphism $\psi\colon E|_U \longrightarrow U \times F$. The bundles of interest in gauge theories are locally trivial.

As mentioned above, the first kind of 'field' that can be defined on a bundle is a *section*. A section of a bundle (E, π, M) is a map $s\colon M \longrightarrow E$ such that the image of each point $p \in M$ lies in the fibre E_p over p. Another way of saying this is that $\pi \circ s = \iota_M$. Sections have especially nice properties when they are defined on a particular kind of bundle called a *vector bundle*.[6] The entire space of sections $\Gamma(E)$ on a vector bundle also has some nice properties. Two of the most important properties of $\Gamma(E)$ are that it is a module over $C^\infty(M)$ and that for any trivial bundle it has a basis e_i so that any $s \in \Gamma(E)$ can be written uniquely as the sum

$$s = s^i e_i.$$

This latter property is especially useful in the case of locally trivial vector bundles since it enables one to work with a basis of sections over some neighborhood of a point in the base space. That sections are so important for gauge field theories becomes particularly evident when it is seen that every section of a vector bundle is actually a vector field. This connection can be pushed further by creating various vector bundle constructions.

Many of the same objects, such as duals, direct sums and tensor products, that can be constructed from vector spaces can be constructed from a vector bundle. Given a vector bundle E one can construct the *dual bundle*

$$E^* = \bigcup_{p \in M} E_p^*,$$

where E_p^* is the dual space of E_p. Given a basis of sections e_i there is a unique dual basis e^i of sections of E^*. In beautiful similarity to the case of sections of a vector bundle, every section of a dual bundle is a 1-form. A natural example of this is the dual bundle of the tangent space TM, which is a vector bundle, called the *cotangent bundle*

$$T^*M \equiv \bigcup_{p \in M} T_p^*M.$$

[6] See definition in Appendix C.

Similarly, given two vector bundles (E, π, M) and (E', π', M) it is possible to construct the *direct sum vector bundle* $E \oplus E'$ over M, where the fibre over $p \in M$ is $E_p \oplus E'_p$, and the *tensor product vector bundle* $E \otimes E'$ over M, where the fibre over $p \in M$ is $E_p \otimes E'_p$. Given a vector bundle E one can also define an *exterior algebra bundle* ΛE where the fibre over $p \in M$ is ΛE_p. The sections of the exterior algebra bundle form an algebra when fitted with a wedge product. Such a construction on the cotangent bundle T^*M generates an object ΛT^*M called the *form bundle.* The differential forms on M are just the sections of the form bundle. Thus we are now able to see that fibre bundles provide a cohesive framework for the treatment of differential forms, which makes it ideal for use in physics since physical laws can be written as differential forms on a manifold. It is also possible to take r copies of the tangent bundle and s copies of the cotangent bundle to form the (r, s) *tensor bundle*

$$TM \otimes \cdots \otimes TM \otimes T^*M \otimes \cdots \otimes T^*M. \tag{5}$$

The sections of this bundle form (r, s) tensor fields. The (r, s) tensor bundles are particularly important in general relativity because the important objects, such as the curvature tensor and the metric tensor, are tensor fields.

There is a special way of constructing a vector bundle from local trivial bundles on a manifold M. This is accomplished by taking an open cover of a manifold M, forming trivial bundles with each of the sets of the cover and some vector space V, and gluing the trivial bundles together to form a vector bundle.[7] The bundles constructed this way are called G-bundles because a group G is required to specify how the pieces connect together. G-bundles are fibre bundles of particular interest in gauge theory because gauge fields in Yang-Mills theories are described as sections of such bundles where the group G is the internal symmetry group of the particular force in question. In the case of electromagnetism, the gauge group is $U(1)$, and thus the $U(1)$-bundle is fundamental to the fibre bundle formulation of electromagnetism.

This finally brings us to the issue of gauge transformations. In order to state the definition precisely we need a few more definitions. If we generalize the notion of an endomorphism to vector bundles we can construct a bundle called the *endomorphism bundle.* Given a vector bundle E over a manifold M, the endomorphism bundle $\mathrm{End}(E)$ is the bundle $E \otimes E^*$. The name is well chosen since the sections of $\mathrm{End}(E)$ are vector bundle morphisms from (E, π, M) to itself. It turns out that each section of $\mathrm{End}(E)$ is a C^∞-linear map $T\colon \Gamma(E) \longrightarrow \Gamma(E)$, mapping sections to sections. The notion of an endomorphism bundle ties into the consideration of gauge transformations in the context of G-bundles since the transformations $T(p)$ that live in the gauge group G or

[7] See Appendix D for a formal summary of this construction.

Lie algebra $\mathfrak{g}$[8] are actually elements of the fibres $\text{End}(E_p)$ of $\text{End}(E)$. Tying things together, T *lives in* $\mathfrak{g}$ if $T(p)$ lives in $\mathfrak{g}$ for all $p \in M$. Now, if (E, π, M) is a G-bundle, where G is some Lie group, and $T \in \text{End}(E)$ then T is a *gauge transformation* if $T(p)$ lives in G for all $p \in M$. The set of all *gauge transformations* forms a group, which we will denote $\mathcal{G}$, to distinguish it from the *gauge group* G of the G-bundle.

2.3 Connection and Curvature

In order to consider derivatives of sections it is necessary to introduce a new object called the connection. Suppose that (E, π, M) is a vector bundle and v is a vector field, then a *connection* on M is a function $D_v \colon \Gamma(E) \longrightarrow \Gamma(E)$, $s \rightsquigarrow D_v s$, that is C^∞-linear in v and real-linear[9] in $s \in \Gamma(E)$ and satisfies the Leibniz law

$$D_v(fs) = v(f)s + fD_v s,$$

for any $v \in \mathbb{V}(M), s \in \Gamma(E)$ and $f \in C^\infty(M)$. $D_v s$ is called the *covariant derivative* of s in the direction of v. If we work locally in some open set $U \subseteq M$ with local coordinates x^μ, the corresponding basis of coordinate vector fields ∂_μ and basis of sections of E over U, then we may obtain the expression

$$D_\mu e_j \equiv D_{\partial_\mu} e_j = A^i_{\mu j} e_i, \tag{6}$$

where the functions $A^i_{\mu j}$ satisfying this relation are called the components of the *vector potential.* The vector potential is useful because it enables one to obtain an expression for the covariant derivative of a section s over U in the direction of v

$$D_v s = v^\mu (\partial_\mu s^i + A^i_{\mu j} s^j) e_i. \tag{7}$$

From the linearity of the last term in equation (7) it can be seen that the vector potential maps a section and a vector field to another section. This enables one to consider the vector potential as a section of the bundle $\text{End}(E|_U) \otimes T^*U$, making the vector potential an *End(E)-valued 1-form.* With this one can define the vector potential A to be

$$A = A^i_{\mu j} e_i \otimes e^j \otimes dx^\mu,$$

and by suppressing the 'internal indicies' i and j the components can be written as

$$A_\mu = A^i_{\mu j} e_i \otimes e^j.$$

[8]The phrase 'live in' is technical and is defined in Appendix C.
[9]Or complex-linear if E is a complex vector bundle

This makes each of the components A_μ a section of $\mathrm{End}(E)$. The power of this way of defining the vector potential is that A can be considered to be an $\mathrm{End}(E)$-valued 1-form on the entire manifold M. This enables us to give an expression for A

$$A = \sum_i T_i \otimes \omega_i,$$

where the T_i are sections of $\mathrm{End}(E)$ and the ω_i are 1-forms on M. From this it can be seen that

$$A(v) = \sum_i \omega_i(v) T_i$$

defines a section of $\mathrm{End}(E)$. It turns out that any connection D can be written as $D^0 + A$:

$$D_v s = (v(s^i) + A^i_{\mu j} v^\mu s^j) e_i = D^0_v s + A(v)s. \tag{8}$$

The connection D^0 is called the *standard flat connection.*

Besides enabling one to differentiate sections, the connection also allows one to move around the vectors in the fibres of vector bundles with minimal change in the direction of the vector. This shifting around of vectors is called *parallel transport.* Suppose (E, π, M) is a vector bundle with a connection D defined on it. Let γ: $[0, T] \longrightarrow M$ be a smooth map from p to q and suppose that for $t \in [0, T]$, $u(t)$ is a vector in the fibre of E over $\gamma(t)$. Then $u(t)$ is *parallel transported* along γ if the following condition holds for all t:

$$D_{\gamma'(t)} u(t) \equiv \frac{d}{dt} u(t) + A(\gamma'(t)) u(t) = 0. \tag{9}$$

The object $D_{\gamma'(t)} u(t)$ is called the *covariant derivative.* This allows one to determine the vector $u(t)$ in $E_{\gamma(t)}$ such that

$$u(0) = u, \qquad D_{\gamma'(t)} u(t) = 0$$

by solving a linear differential equation.

A companion object to the connection is an object called the *curvature*, which is a measure of the failure of covariant derivatives to commute. Given a vector bundle (E, π, M), two vector fields v and w on M and a connection D, the curvature $F(v, w)$ is defined to be the operator on sections of E defined by

$$F(v, w)s = D_v D_w s - D_w D_v s - D_{[v,w]} s,$$

where the last term is included to correct for the fact that the covariant derivatives may fail to commute because the vector fields v and w fail to commute, i.e. have non-vanishing Lie bracket. The curvature has the property that it is C^∞-linear over v, w and

s. A connection with curvature zero, $F(v, w) = 0$ for any v, w and s, is a *flat conneciton*. Analogous to the case of the connection, it is possible to consider the curvature to be an End(E)-valued 2-form. If we let $F_{\mu\nu}$ be the section of End(E) given by

$$F_{\mu\nu} = F(\partial_\mu, \partial_\nu),$$

then $F(v, w) = v^\mu w^\nu F^{\mu\nu}$ by linearity so that if $e_j \otimes e^i$ is a local basis of sections for End(E), then we have

$$F_{\mu\nu} = F^j_{\mu\nu i} e_j \otimes e^i.$$

The functions $F^j_{\mu\nu i}$ are the components of the curvature. This is made concrete by defining the *curvature 2-form* F to be an End(E)-valued 2-form

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu,$$

a section of the bundle End$(E) \otimes \Lambda T^* M$.

3 Fibre Bundles in Philosophy

The theory of fibre bundles enables the formulation of the theories of each of the four known fundamental forces of nature. The theories of the strong, weak and electromagetic fields are quantized Yang-Mills theories, which can be formulated as quantized versions of the corresponding classical Yang-Mills theories. The classical Yang-Mills theories[10] fundamentally involve G-bundles, where G is the gauge group of the theory in question: $U(1)$ for electromagnetism; $U(1) \times SU(2)$ for electroweak theory; and $SU(3)$ for the strong theory. The quantized versions of these three theories (QED, GWS electroweak theory and QCD) form the main part of the standard model of particle physics. General relativity, though a very different sort of theory, can also be formulated using fibre bundles but the fundamental vector bundle is the tangent bundle of the spacetime manifold.[11]

The rest of this paper is concerned with a survey of some philosophical issues in order to examine the role that the fibre bundle formalism plays in the clarification of some of the philosophical, particularly ontological, issues that are raised by the standard model and general relativity and in the interplay between physics and mathematics.

[10] Any consideration of Yang-Mills theories in this paper will be resricted to the classical versions.

[11] It is possible to set up general relativity using a G-bundle as well, where the total space is the bundle of frames $\mathbf{B}(M)$, $G = GL(n, \mathbb{R})$ and the fibre $F = \mathbb{R}^n$ but we will not consider this formulation here.

3.1 Ontology of Gauge Potentials

In classical electromagnetism the electromagnetic field can be calculated from a gague potential called the 4-vector potential $A_\mu = (V, \vec{A})$, where V is the scalar potential and $\vec{A}$ is the vector potential,[12] which are related to the electric (vector) field $\vec{E}$ and the magnetic (pseudo-vector) field $\vec{B}$ by $\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} V$ and $\vec{B} = \vec{\nabla} \times \vec{A}$. The gauge field corresponding to A_μ is the electromagnetic tensor field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, which has as its components the components of the electric and magnetic fields. The striking feature of A_μ is that $F_{\mu\nu}$ is invariant under the gauge transformation

$$A_\mu \longrightarrow A'_\mu = A_\mu - \partial_\mu \Lambda,$$

for some differentiable scalar function Λ, which is what makes A_μ a gauge potential.

For classical particle dynamics, it is $F_{\mu\nu}$ that determines the evolution of a particle in an electromagnetic field and A_μ is just a convenient means for doing calculations. For the quantum mechanical treatment of a particle, however, it is the 4-vector potential that appears in the dynamical equation (Schrödinger's equation) and this as it turns out, causes A_μ to have observable consequences.[13] The effect that can be attributed to the 4-vector potential is, of course, the Aharonov-Bohm effect. The effect is produced in a two slit experiment in which a solenoid is inserted between the slits on the screen side of the experiment. It turns out that there is an observable shift in the interference pattern on the screen due to the non-zero magnetic field from the solenoid, even though the field value is, or can be made to be, zero outside of the solenoid. This observable effect has been attributed to the 4-vector potential, barring some action at a distance of the electromagnetic field, which implies that A_μ is somehow physically realized and is not just a calculational aid.

The aspect of this that makes giving a physical interpretation to the 4-vector field A_μ so difficult is that the shift in the interference pattern, as well as $F_{\mu\nu}$ of course, is identical for any A'_μ obtained by a gauge transformation from A_μ. This makes it difficult to suppose that each A_μ corresponds to something physically real since this would imply a radical underdetermination in the theory. Given the character of the situation it seems more natural to consider the physically real entity to be an element of the space of 4-vector potentials modulo gauge transformations, so that each A_μ is

[12] The vector potential $\vec{A}$, perhaps more appropriately called the 3-vector potential, should not be confused with the vector potential A_μ from the previous section. As the notation suggests, the 4-vector potential A_μ, however, *is* a special case of the vector potential A_μ from the previous section.

[13] Note that this does not require QED since we can consider a quantum mechanical particle moving in a classical electromagnetic field.

really just a representative of an equivalence class. This of course requires a deeper examination. Before we take these ontological questions any further it is necessary to bring the other gauge fields into the mix.[14]

The classical Yang-Mills theories are all generalizations of electromagnetism and, consequently, share much of the same general structure. One of the most important structural features of this class of theory is a *gauge potential* from which the corresponding *gauge field* can be obtained by covariant differentiation. The theory of fibre bundles provides a nice way of making the similarities clear. The fundamental bundle for such a gauge field theory is a G-bundle (E, π, M), where the Lie group G acts freely on the total space E. Such a bundle is called a *principal fibre bundle* with fibre G, the *structure group* of the bundle. In the case of Yang-Mills theories M is the spacetime manifold or some object mathematically related to it. The gauge potential[15] is determined by a G-connection on this principal fibre bundle. The vector potential is the quantity A when represented as in equation (8). Given a particular section of E, which corresponds to a specification of the gauge, the vector potential can be locally expressed uniquely in the familiar form A_μ as described in the previous section by pulling back the one-form field onto the manifold M. The gauge field[16] is then represented by the curvature of the vector potential. Similarly the gauge field can be given the usual local coordinate representation $F_{\mu\nu}$, a 2-form field on M, by pulling back the curvature.

Depending on the topological characteristics of the bundle, sections may only be definable locally. In such a case the terms 'section' and 'gauge transformation' must be predicated by the term 'local.' It turns out that the lack of 'magnetic monopoles' in the given Yang-Mills theory obviates this restriction, allowing the definition of global sections and gauge transfomations. As is reflected by the divergence free magnetic field in Maxwell's equations, there do not appear to be any 'electromagnetic' magnetic monopoles, so that the vector potential can be given a global description as a 1-form field on M.

The fibre bundle formalism also makes plain the connection between the principal fibre bundle for the theory and other associated vector bundles[17] derived from it that describe other aspects of a given system. An example of this that is relevant to the present case is that a quantum mechanical particle can be represented by an associated vector bundle that is a G-bundle where the fibre is not the gauge group G itself but a representation

[14] The exposition to follow owes much to Healey's paper [5].

[15] The usual electromagetic 4-vector potential in the case of electromagnetism

[16] The electromagnetic field in the case of electromagnetism

[17] It is possible to generate new vector bundles from a principal fibre bundle using the action of G, see [6] pp. 135-36. Such bundles are called *associated vector bundles*.

of it and the base space is the same spacetime manifold M. The wave-function of a particle in the position representation is given by a section of this vector bundle. The value of the section for a particular $p \in M$ is the phase of the wave-function in that position representation. Interestingly, there is a one to one correspondence between sections of the principal bundle and sections of the associated bundle. This implies that a gauge transformation on the principal bundle induces a gauge transformation on the associated bundle. The gauge transformations on both bundles leave the physics of both the particle and the electromagnetic field invariant, so 'gauge transformation' is an appropriate term for both. The fact that the associated gauge transformations of the gauge potential and the wave-function come out of the mathematics rather than having to be imposed seems to be a strength of the fibre bundle formulation.

Thus, it is clear that the fibre bundle formalism makes plain the connection between the gauge fixing of the gauge potential and the phase of the wave-function of a particle evolving in the presence of the corresponding gauge field. It is also instructive because it gives a coordinate free representation of all of the physical objects involved. Despite these advantages, however, the choice of a connection on the principal fibre bundle does not uniquely specify the gauge field and the phase of the wave-function. Cerain G-connections of the principal fibre bundle that are related by a gauge transformation g from the group $\mathcal{G}$, described in the previous section, leave the physics invariant. G-connections related in this way are called *gauge-equivalent*, Thus the gauge potential A is not a good candidate for a physically real object since it retains the same underdetermination of A_μ mentioned above. This is so since there is no evidence that gauge-equivalent quantities are experimentally distinguishable. Thus, we seek some kind of gauge-invariant quantity that could represent the 'physical' gauge potential, *viz.*, a mathematical object that represents the phenomenon that gives rise to the phase shift in the Aharonov-Bohm effect (or its equivalent for other gauge fields).

Following the intuition described above, the fibre bundle formalism naturally suggests a corresponding candidate. Denoting the space of G-connections on the principle fibre bundle by $\mathcal{A}$, the quotient $\mathcal{A}/\mathcal{G}$, called the space of connections modulo gauge transformations, is this candidate space of gauge-invariant quantities. Thus, then the 'physical' gauge potential would correspond to an element of $\mathcal{A}/\mathcal{G}$. Such a construction is more natural in the case of the fibre bundle theory since $\mathcal{A}/\mathcal{G}$ is an object of independent interest in gauge theory. Healey [5], however, suggests another gauge-invariant quantity to correspond to the 'physical' gauge potential, namely *Wilson loops*.[18] This is a sensible

[18] A Wilson loop is formed by taking the trace of the holonomy around a loop in M. The holonomy of a curve from p to q in M given some connection D is a map from E_p to E_q taking each vector in E_p to its parallel transported counterpart in E_q. In the case where the curve is a loop, the holonomy is a

suggestion since in the case of electromagnetism, a $U(1)$ gauge theory, the Wilson loop is simply the phase acquired by a charged particle as it moves around a loop in the vector potential corresponding to the G-connection and this is precisely the sort of phase that gives rise to interference effects like the Aharonov-Bohm effect [2]. Interestingly, for abelian gauge theories, such as electromagnetism, the holonomy itself is gauge-invariant but for non-abelian gauge theories only the Wilson loop is gauge-invariant [2]. A further discussion of this is beyond the scope of this paper, so I will move on to discuss how fibre bundles tie into ontological issues in the case of general relativity.

3.2 Relationism versus Substantivalism in General Relativity

The debate between substantivalists, who consider spacetime (or just space) to be ontologically prior to matter-energy, and relationists, who give ontological priority to spatiotemporal relations between material entities, has been going on since the time of Newton and Liebniz. The character of the recent debate bears little similarity to the original one between Newton and Leibniz, however, because since the development of gauge theories, especially electromagnetism—the original gauge theory, it has become necessary to consider matter-energy fields rather than particles, and because the geometric properties of spacetime itself have been attached to a special field, namely the metric field, since the development of general relativity. The consideration of the metric as another sort of field comes from the fact that it, as all gauge fields do, contains energy-momentum,[19] which implies that it should be considered as a 'material' object contained within spacetime and should not be considered part of spacetime itself. Thus, the recent tradition, notably Earman [3] and Earman and Norton [4], has been to consider the bare manifold M to be spacetime. Then the point of debate is whether or not the spacetime manifold is substantival.

In the wake of the collapse of logical positivism there was a revitalization of scientific realism among philosophers, including realism about spacetime. The weight of support has now switched over to the relationsim side as a result of the (re-)discovery of arguments that challenge substantivalism. The most prominent such argument is a modern form of Einstein's 'hole argument' given by Earman and Norton [4]. The argument uses a model theoretic approach to show that manifold substantivalism leads to a pernicious form of indeterminism for a large class of spacetime theories. The argument proceeds as follows. First of all, a model of a local spacetime theory consists of an $(n+1)$-tuple $(M, O_1, \ldots, O_n)$, where M is a manifold and the O_i comprise a set n geometric objects, subject to the following condition: a subset of the O_i vanish, giving the field equations

map from E_p to itself, where p is the starting point of the loop.

[19] Particularly in the form of gravitational waves, predicted by general relativity, which can exist in a universe in which all gauge fields are zero everywhere.

for the theory, i.e.

$$O_k = 0; O_{k+1} = 0; \cdots ; O_n = 0,$$

and the objects in the field equations are tensors. This condition implies that for any model $\mathfrak{M} = (M, O_1, \ldots, O_n)$ of the spacetime theory and a diffeomorphism h of M, the $(n+1)$-tuple $(M, h \cdot O_1, \ldots, h \cdot O_n)$ is also a model, call it $\mathfrak{M}'$, because the field equations are tensor equations. This leads directly to the hole argument. Suppose $\mathfrak{M}$ is a model of a local spacetime theory $\mathcal{T}$ with manifold M and let H be any open set of M (the hole). Then because there are arbitrarily many diffeomorphisms of M that are the identity on $M \setminus H$ and differ smoothly from the indentity on H, there are arbitrarily many *distinct* models $\mathfrak{M}^i$ of $\mathcal{T}$ that differ from one another only in the hole H.[20] Since general relativity is a local spacetime theory of this type, the hole argument applies to general relativity.

Now, for the manifold substantivalist, each of the models $\mathfrak{M}^i$ represent physically distinct spacetimes because the diffeomorphism that takes $\mathfrak{M}$ to $\mathfrak{M}^i$ changes the fields on M within the hole H so that the same points on the manifold M are assigned different field values. Since the manifold is considered to be physically real, a different field assignment constitutes a different physical situation. The indeterminism then arises for the manifold substantivalist due to the fact that the evolution of fields is not determined by their specification on any set of spacelike hypersurfaces S if the hole appears in the future of each of the elements of S. In fact, as Earman and Norton [4] mention, even a local specification of a spacelike hypersurface together with some boundary condition produces the same indeterminism if the hole is in the future of the hypersurface and the interior of the boundary. Furthermore the substantivalist commits herself to other problems since the only physical observables are point-coincidences and such coincidences are preserved by diffeomorphisms, so the substantivalist is then committed to physically different spacetimes $\mathfrak{M}^i$ that are empirically indistinguishable. These two problems together, Earman and Norton argue, make manifold substantivalism an untenable position since the substantivalist is committed *a priori* to indeterminism, which is unacceptable because if determinism fails, then 'it should fail for a reason of physics, not because of a commitment to substantival properties which can be eradicated without affecting the empirical consequences of the theory' [4].

This provides a natural opening for the relationist, who denies the existence of the spacetime manifold M. The spacetime models $\mathfrak{M}^i$ preserve spatiotemporal relations and so the relationist can argue that they represent the same physical situation. Thus, for the relationist, diffeomorphism invariance is a non-physical symmetry of the theory,

[20]Consider i to be an element of some index set I for all such models.

something reminiscent of gauge freedom. It turns out that the theory of fibre bundles enables a clarification of the status of diffeomorphism invariance as a gauge invariance in general relativity.

The fundamental objects in general relativity are the spacetime manfold M and the Lorentzian spacetime metric g. The pair (M, g) is a pseudo-Riemannian manfiold,[21] which we will refer to simply as the spacetime manifold M. The metric g is a symmetric C^∞-linear bilinear form on $\mathbb{V}(M)$, the space of vector fields on M. This enables one to consider g to be a $(0, 2)$ tensor field, which can be considered as a section of the $(0, 2)$ tensor bundle $T^*M \otimes T^*M$, see (5). Given a set of local coordinates, we can write the metric in the component form $g_{\mu\nu}$.

It is tempting to follow the lead of the previous section and look for a way of dividing metrics into equivalence classes of physically equivalent metrics by constructing orbits of sections of some subbundle of the $(0, 2)$ tensor bundle[22] under the action of the diffeomorphism group Diff(M) on M. It turns out that this can be done, *viz.* one can form the quotient Riem(M)/Diff(M) of Lorentzian metrics modulo diffeomorphisms. This turns out to be problematic, however, because the 'Gribov-Singer effect means that there is an intrinsic obstruction to choosing a gauge that works for all physical configurations;' Isham goes on to mention that 'whether or not this has any real significance in the theory is still a matter of debate.' [6], 134. Nevertheless, the relationist view leads one to take view general relativity as a gauge theory, with Diff(M) as the space of gauge transformations. Moreover, this perspective is given concreteness by the fibre bundle formalism which enables the construction of a principal bundle with Diff(M) as the structure group that has the property that the quotient space Riem(M)/Diff(M) gives the physically distinct metrics on M.

3.3 The Generality of Fibre Bundles

The power of the theory of fibre bundles seems to derive from its great generality, but with this generality also comes the unification of a large variety of mathematical and physical systems. It is this aspect of the fibre bundle theory that should make it of interest not only to philosophers of physics but also philosophers of mathematics. This section is devoted to an attempt give an indication of how remarkable the ability of the fibre bundle formalism to unify concepts actually is.

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[21] The prefix 'pseudo' applies because the metric is not positive semi-definite. Specifically the metric g is Lorentzian, i.e. has signiture $(1, 3)$.

[22] It would need to be a subbundle of the $(0, 2)$ tensor bundle because not all sections of the $(0, 2)$ tensor bundle are metrics. This is so since not all sections have the requisite symmetry property.

The fibre bundle formalism is built upon the theory of differential geometry on manifolds. As a result of this it incorporates the features and strengths of differential geometry but it does so in a very natural way. Differential geometry itself generalizes vector calculus on $\mathbb{R}^n$, providing a framework essential to modern physics and mathematics. One of the indications of the power of the generalization to differential geometry is the ability to express Maxwell's equations in coordinate free form. As mentioned in section 3.1, the electromagnetic field is written as a 2-form field F on some manifold M. In the case that M is Minkowski space F can be split up into electric and magnetic fields,

$$F = B + E \wedge dt, \tag{10}$$

where the magnetic field B is the 2-form $B = B_x dy \wedge dx + B_y dz \wedge dx + B_z dx \wedge dy$, and the electric field E is the 1-form $E = E_x dx + E_y dy + E_z dz$ and t is the coordinate time. From the properties of the exterior derivative operator d, the divergence is just the exterior derivative of 2-forms on $\mathbb{R}^3$ and the curl is the exterior derivative of 1-forms on $\mathbb{R}^3$ which means that the first two Maxwell equations,

$$\nabla \cdot \vec{B} = 0,\ \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0,$$

can be written as

$$dF = 0. \tag{11}$$

By employing the Hodge star operator $\star$: $\Omega^p(M) \longrightarrow \Omega^{n-p}(M)$, for n-dimensional manifold M (in the present resricted case $M = \mathbb{R}^3$ for the vector fields) the last two of Maxwell's equations,

$$\nabla \cdot \vec{E} = \rho,\ \nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{j},$$

can be written as

$$\star d \star F = J, \tag{12}$$

where $J = j - \rho dt$, is the current written as a 1-form.[23] The remarkable aspect of equations (11) and (12) are that they now hold for any 4-manifold M where it is not required that F can be split as in equation (10). This approach is also interesting because it incorporates the fact that $\vec{B}$ is a pseudo-vector field because B is invariant under a parity transformation by virtue of the fact that it is a 2-form.

Now, all of this can be done using just coordinate free differential geometry but the concepts are more clear with a shift to the fibre bundle formalism. A differential form is understood as a section of the form bundle $\Lambda T^* M$. The p-forms are just sections of the

[23]Here $\vec{j} = j^1\partial_1 + j^2\partial_2 + j^3\partial_3$, which can be converted into the 1-form is $j = j_1 dx^1 + j_2 dx^2 + j_3 dx^3$.

subbundles $\Lambda^p T^*M$ of the form bundle. In the fibre bundle formalism this is naturally generalized to the case of the *exterior algebra bundle* ΛE for any vector bundle E. It can then be shown that the exterior derivative and the Hodge star operator can be generalized to act on End(E)-valued differential forms. Then when supplied with a connection D on the bundle E, this determines a exterior derivative d_D on End(E)-valued differential forms which, for any End(E)-valued 2-forms, such as the gauge field, the Bianchi identity

$$d_D F = 0 \tag{13}$$

holds. This is the Yang-Mills generalization of (11). Then, with the appropriate generalization of $\star$,[24] and any End(E)-valued 1-form J on M, called the current, we obtain the Yang-Mills equation

$$\star d_D \star F = J, \tag{14}$$

the generalization of (12). Equations (13) and (14) are the field equations for classical Yang-Mills theories discussed in section 3.1, which are naturally formulated in the theory of fibre bundles.

Thus, we see that differential geometry generalizes calculus on $\mathbb{R}^n$ to general smooth manifolds, but that the theory of fibre bundles generalizes differential geometry and provides a general framework that clearly expresses the structural similarities between the classical versions of the theories of the standard model. The quantized versions of the Yang-Mills theories can then be obtained from the Yang-Mills Lagrangian by path-integral quantization of the Yang-Mills action [2]. Moreover, the structure of the fibre bundle formalism elucidates connections between different phenomena, as was seen in section 3.1 in the conneciton of the principal bundle to the associated bundle describing the quantum mechanical particle. This is the sense in which the fibre bundle formalism seems to not only have great mathematical generality, but also a strong capacity for conceptual unification of mathematical and physical systems. This seems to derive from the fact that the fibre bundle formalism uses a relatively small number of extremely flexible concepts, such as section, connection and curvature and the notion of a fibre bundle itself.

The generality and flexibility of these concepts is demonstrated nicely by the formulation of general relativity using fibre bundles. As mentioned in the opening of section 3, the fundamental bundle in general relativity is the tangent bundle of the spacetime manifold M. It turns out that the choice of metric g[25] on M gives rise to a unique

[24]See [2] p. 261.
[25]Or $g_{\alpha\beta}$ in local coordinates

connection on the tangent bundle that is metric preserving and torsion free[26], called the *Levi-Civita* connection ∇. In analogy to the case of a vector potential, given a local basis of coordinate vector fields on M the *Christoffel Symbols* $\Gamma^{\gamma}_{\alpha\beta}$ for the Levi-Civita connection are given by

$$\nabla_{\alpha}\partial_{\beta} = \Gamma^{\gamma}_{\alpha\beta}\partial_{\gamma}.$$

The equation for parallel transport naturally adapts from the general theory of fibre bundles. This can be used to obtain the familiar geodesic equation

$$\frac{d^2\gamma^{\delta}}{dt^2} + \Gamma^{\delta}_{\alpha\beta}\frac{d\gamma^{\alpha}}{dt}\frac{d\gamma^{\beta}}{dt}, \tag{15}$$

which specifies the kinematics. The curvature of the Levi-Civita connection just turns out to be the *Riemann curvature tensor* $R(u, v)$:

$$R(u, v)w = (\nabla_u\nabla_v - \nabla_v\nabla_u - \nabla_{[u,v]})w.$$

Given a local basis of vector fields the Riemann tensor can be expressed in the usual way, *viz.* by its coordinates $R^{\alpha}_{\beta\gamma\delta}$. Then in the usual way the contraction $R_{\alpha\beta} = R^{\gamma}_{\alpha\gamma\beta}$ is the *Ricci tensor* and the contraction $R = g^{\alpha\beta}R_{\alpha\beta} = R^{\alpha}_{\alpha}$ is the *Ricci scalar.* Then if we let $T_{\alpha\beta}$ be the sress-energy tensor, then we may write the Einstein equations as

$$R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = 8\pi T_{\alpha\beta}.$$

Thus we see that with general relativity, as with Yang-Mills theories, the choice of the appropriate vector bundle, the general theory of sections, connections and curvature enables the theory to be set up in a conceptually economical and clear way. It is particularly nice in the case of general relativity since, with two natural restrictions, the connection on the tangent bundle is unique and turns out to be the standard connection used in the differential geometric formulation of GR and its curvature is the fundamental Riemann tensor.

This capacity for clear exposition of ideas seems to be another of the great advantages of the fibre bundle formalism. Contemporary theories in physics are extrordinarily complex conceptually and a conceptual framework that naturally brings out the important structural features and connections between different parts of a theory and among different theories is extremely advantageous. This seems to give the fibre formalism a genuine epistemological advantage over less general mathematical frameworks. It is not yet clear

[26] The metric preserving property means that tangent vectors do not change length when parallel translated, just the condition we require. The torsion free property means that tangent vectors do not rotate when parallel transported [2], but the details of this are not important to us here.

whether the fibre bundle formalism reflects a something deep about the structure of the physical world but it is intriguing that all of the fundamental theories of modern physics admit of a similar treatment within it.[27] Of course the fibre bundle theory does not enable general relativity to be expressed in the same form as Yang-Mills theories, but it does make certain similarities and differences plain. Perhaps insight into the origin of these similarities and differences will lead to a development of a reconciliation of general relativity and the standard model. In any case, an examination of these similarities and differences between the two in a *single* mathematical framework is sure to be instructive.

4 Conclusion

We have seen that the fibre bundle theory provides a cohesive framework for the treatment of the fundamental theories of modern physics, has great power inherent in it due to its generality and helps to clarify the treatment of the ontology of gauge potentials and the metric field. Its abilty to formulate theories makes it an indispensible tool for philosophers of physics, who must understand the structure of physical theories in order to examine what they say about the world, because the formalism makes the structural characteristics of theories clear, at least in the classical case that we have considered here. Its ability to generalize mathematical concepts and provide a uniform treatment for various physical theories should also make it of interest to philosophers of mathematics.

The fibre bundle formalism is clearly relevant to the philosophy of contemporary physics and mathematics. Indeed, the discussion in this paper has only scratched the surface of this deep and intriguing mathematical theory. The power and flexibility of the theory also suggests that it will be useful to philosophers interested in the history and philosophy of physics and mathematics since it could provide a useful framework for comparing different theories in the historical development. In sum, the above discussion serves to illustrate that the theory of fibre bundles is sure to be an object of lasting interest and utility to philosophers of physics and mathematics.

[27] Especially given the formulation of general relativity using the principal bundle of frames, see footnote 11.

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A Definitions from Differential Geometry

A.1 Manifolds

topological space: a set X together with a family of subsets of X, called the open sets, which satisfy the following conditions:

1. The empty set and X are open sets
2. If $U, V \subseteq X$ are open, so is $U \cap V$.
3. If the sets $U_\alpha \subseteq X$ are open, so is the union $\bigcup U_\alpha$.

topology: the topology of X is the set of its open subsets.

neighborhood: an open set containing a point $x \in X$ is a neighborhood of x.

closed set: A set which is the complement of an open set.

continuous function: a function $f\colon X \longrightarrow Y$ is continuous if, given any open set $U \subseteq Y$, $f^{-1}(U) \subseteq X$ is open.

homeomorphism: a continuous function with a continuous inverse.

chart: given a toplogical space X and an open set $U \subseteq X$, a chart is a homeomorphism $\varphi\colon U \longrightarrow \mathbb{R}^n$, where the inverse is defined on $\varphi(U)$.

n-dimensional manifold: a topological space M together with charts $\varphi_\alpha\colon U_\alpha \longrightarrow \mathbb{R}^n$, where U_α are open sets covering M, such that the **transition function** $\varphi_\alpha \circ \varphi_\beta^{-1}$ is smooth where defined.

A.2 Vector Fields

Algebra: a vector space V together with a law of composition, called a product, of two vectors that turns V into a ring. A commutative algebra is an algebra with a commutative product.

Module: Let R be a commutative ring. An R-module V is an abelian group with law of composition $+$, together with a scalar multiplication $R \times V \longrightarrow V$, written $(r, v) \rightsquigarrow rv$, which satisfies the following axioms:

1. $1v = v$,
2. $(rs)v = r(sv)$,
3. $(r + s)v = rs + rv$,
4. $r(v + v') = rv + rv'$,

for all $r, s \in R$ and $v, v' \in V$. Note that an F-module, where F is a field, is an F-vector space. A *free module* is an R-module that is isomorphic to R^k for some $k \in \mathbb{N}$.

$C^\infty(M)$**:** the commutative algebra of infinitely differentiable real valued functions on M.

vector field: a vector field v is a linear function $v\colon C^\infty(M) \longrightarrow C^\infty(M)$ that satisfies the Leibniz property:

$$v(fg) = v(f)g + fv(g).$$

$\mathbb{V}(M)$**:** the set of vector fields on M. With the addition operation

$$v + w \equiv (v + w)(f) = v(f) + w(f),$$

and the 'scalar multiplication' operation

$$gv \equiv (gv)(f) = gv(f),$$

$\mathbb{V}(M)$ is a free module over $C^\infty(M)$.

tangent vector: suppose $p \in M$, then a tangent vector at p is a linear function $v_p\colon C^\infty(M) \longrightarrow \mathbb{R}$ that satisifes a Leibniz rule

$$v_p(fg) = v_p(f)g(p) + f(p)v_p(g).$$

Note that given $v \in \mathbb{V}(M)$, $v_p(f) \equiv v(f)(p)$ is a tangent vector at p.

tangent space at p**:** the set $T_p(M)$ of all tangent vectors at $p \in M$. Using the natural definitions (similar to above) of addition and scalar multiplication, $T_p(M)$ is a real vector space.

curve: if I is an interval of $\mathbb{R}$, then a curve is a smooth function $\gamma\colon I \longrightarrow M$, i.e. for any $f \in C^\infty(M), f(\gamma(t))$ depends smoothly on t. The tangent vector $\gamma'(t)$ is the function $\gamma'(t)\colon C^\infty(M) \longrightarrow \mathbb{R}$ defined by $f \rightsquigarrow \frac{d}{dt} f(\gamma(t))$.

pullback: let M and N be manifolds and let $\varphi : M \longrightarrow N$ be a function. If $f\colon N \longrightarrow \mathbb{R}$ is a function, then the pullback of f is a function $\varphi^* f\colon M \longrightarrow \mathbb{R}$ given by

$$\varphi^* f = f \circ \varphi.$$

(smooth) map: a function $\varphi\colon M \longrightarrow N$ is smooth if $f \in C^\infty(N)$ implies that $\varphi^* f \in C^\infty(M)$.

pulling back: given a map $\varphi\colon M \longrightarrow N$, pulling back is an operation

$$\varphi^*\colon C^\infty(N) \longrightarrow C^\infty(M).$$

pushforward: let M and N be manifolds and let $\varphi\colon M \longrightarrow N$ be a function. If $v \in T_p(M)$, then the pushforward of v by φ is a function $\varphi_* v\colon C^\infty(N) \longrightarrow \mathbb{R}$ given by

$$\varphi_* v = v \circ \varphi^*,$$

where $\varphi_* v \in T_{\varphi(p)}(N)$.

pushing forward: given a map $\varphi\colon M \longrightarrow N$ and a point $p \in M$, pushing forward is an operation

$$\varphi_*\colon T_p(M) \longrightarrow T_{\varphi(p)}(N).$$

integral curve: let $v \in \mathbb{V}(M)$ and $p \in M$. The integral curve through p of the vector field v is the solution $\gamma(t)$ to the initial value problem

$$\gamma'(t) = v_{\gamma(t)}; \quad \gamma(0) = p.$$

integrable vector field: a vector field for which all the integral curves are defined for all t.

flow: let v be an integrable vector field. The flow generated by v is the family of smooth maps $\{\varphi_t\}$, $\varphi_t\colon M \longrightarrow M$, obtained as solutions to the equation

$$\frac{d}{dt}\varphi_t(p) = v_{\varphi_t(p)}.$$

Lie Bracket (commutator): if $v, w \in \mathbb{V}(M)$, the Lie bracket $[v, w]$ is defined by

$$[v, w] = vw - wv.$$

The Lie Bracket is a vector field on M.

A.3 Differential Forms

1-form: if M is a manifold, then a 1-form is a linear (over $C^\infty(M)$) map from $\mathbb{V}(M)$ to $C^\infty(M)$.

$\Omega^1(M)$: the space of all 1-forms on a manifold M. $\Omega^1(M)$ is a free module over $C^\infty(M)$.

exterior derivative: the exterior derivative (or differential) of f is the 1-form df defined by

$$df(v) = vf.$$

differential: the linear map d: $C^\infty(M) \longrightarrow \Omega^1(M)$ defined by $f \rightsquigarrow df$. The differential satisfies a Leibniz law:

$$d(fg) = fdg + gdf.$$

cotangent vector: let M be a manifold and let $p \in M$. A cotangent vector ω at p is a linear map ω: $T_p(M) \longrightarrow \mathbb{R}$. Note that given $\omega \in \Omega^1(M)$ and $v \in \mathbb{V}(M)$, $\omega_p(v_p) \equiv \omega(v)(p)$ is a cotangent vector at p.

cotangent space at p: the set $T_p^*(M)$ of all cotangent vectors at p. Using natural definition of addition and scalar multiplication, $T_p^*(M)$ is a vector space.

dual of f: gien a linear function f: $V \longrightarrow W$, where V and W are vector spaces, a dual function f^*: $W^* \longrightarrow V^*$ defined by $(f^*\omega)(v) \equiv \omega(f(v))$.

pullback (of cotangent vector): let φ: $M \longrightarrow N$ be a mapping between manifolds, with $\varphi(p) = q$. Then the pullback of $\omega \in T_q(N)$ by φ is the function $\varphi^*\omega$: $T_q^*(N) \longrightarrow T_p^*(M)$ defined by

$$(\varphi^*\omega)(v) = \omega(\varphi_* v).$$

pullback (of a 1-form): The pullback of $\omega \in \Omega^1(N)$ is the 1-form $\varphi^*\omega \in \Omega^1(M)$, where the map is defined by

$$(\varphi^*\omega)_p = \varphi^*(\omega_q).$$

natuarality of the differential: given a function f on N and a map φ: $M \longrightarrow N$, we have

$$\varphi^*(df) = d(\varphi^* f).$$

local coordinates: let φ: $U \longrightarrow \mathbb{R}^n$ be a chart, then the pullback of the coordinate functions x_μ on $\mathbb{R}^n$ to U by φ gives a set of local coordinates on U denoted simply by x_μ.

coordinate vector fields: let φ: $U \longrightarrow \mathbb{R}^n$ be a chart, then the pushforward of the coordinate vector fields ∂_μ by φ^{-1} form a basis for vector fields on U. These basis elements, denoted simply by ∂_μ, are the coordinate vector fields associated to the local coordinates x_μ on U. Thus one writes vector fields on U as

$$v = v^\mu \partial_\mu \equiv \varphi^* v^\mu {\varphi^{-1}}_* \partial_\mu.$$

coordinate 1-forms: let φ: $U \longrightarrow \mathbb{R}^n$ be a chart, then the pullback of the coordinate 1-forms dx^μ by φ form a basis for 1-forms on U. These basis elements, denoted simply by dx^μ, are the coordinate 1-forms associated to the local coordinates x_μ on U. Thus one writes 1-forms on U as

$$\omega = \omega_\mu dx^\mu \equiv \varphi^* \omega_\mu \varphi^* dx^\mu.$$

change of basis (vector fields): the change of basis formula for vector fields is

$$\partial_\mu = \frac{\partial x'^\nu}{\partial x^\mu} \partial'_\nu,$$

which in terms of coordinates gives

$$v'^\nu = \frac{\partial x'^\nu}{\partial x^\mu} v^\mu,$$

change of basis (1-forms): the change of basis formula for 1-forms is

$$dx'^\nu = \frac{\partial x'^\nu}{\partial x^\mu} dx^\mu,$$

which in terms of coordinates gives

$$\omega'_\nu = \frac{\partial x^\mu}{\partial x'^\nu} \omega_\mu,$$

dual basis: given a general basis e_μ for vector fields on a chart U, where $e_\mu = T^\nu_\mu \partial_\mu$ for some invertible T^ν_μ, there exists a unique dual basis of 1-forms f^μ on U such that $f^\mu(e_\nu) = \delta^\mu_\nu$.

exterior algebra: let V be a vector space. The exterior algebra, denoted ΛV, is the algebra generated by wedge products of elements of V, i.e. $v \wedge w$ for $v, w \in V$, subject to the constraint $v \wedge w = -w \wedge v$.

$\Lambda^p V$: the subspace of ΛV consisting of linear combinations of p-fold products of vectors in V, e.g. $v_1 \wedge \cdots \wedge v_p$. Elements of V in this set are said to have degree p. $\Lambda^0 V$ is defined to be $\mathbb{R}$. The dimension of $\Lambda^p V$ is $\binom{n}{p}$ if V is an n-dimensional space and $\Lambda^p V$ is empty if $p > n$. It also happens to be the case that

$$\Lambda V = \bigoplus \Lambda^p V$$

and that ΛV is 2^n dimensional.

differential forms: the differential forms on a manifold M is the algebra, denoted $\Omega(M)$, generated by wedge products of elements of $\Omega^1(M)$ subject to the constraint $\omega \wedge \mu = -\mu \wedge \omega$ for each $\omega, \mu \in \Omega^1(M)$. Locally finte linear combinations, i.e. linear combinations for which for every point p in M has a neighborhood where only finitely manyy terms are nonzero.

$\Omega^p(M)$: the space of all linear combinations of p-forms, i.e. products of p 1-forms. With this we have that

$$\Omega(M) = \bigoplus_p \Omega^p(M).$$

exterior derivative (differential): the unique set of maps

$$d\colon \Omega^p(M) \longrightarrow \Omega^{p+1}(M)$$

such that the following properties hold:

1. $d : \Omega^p(M) \longrightarrow \Omega^{p+1}(M)$ agrees with the previous definition.
2. $d(\omega + \mu) = d\omega + d\mu$ and $d(c\omega) = cd\omega$ for all $\omega, \mu \in \Omega(M)$ and $c \in \mathbb{R}$.
3. $d(\omega \wedge \mu) = d\omega \wedge \mu + (-1)^p \omega \wedge d\mu$ for all $\omega \in \Omega^p(M)$ and $\mu \in \Omega(M)$.
4. $d(d\omega) = 0$ for all $\omega \in \Omega(M)$.

The differential has the very important property that $d^2 = 0$.

natuarality of the differential: given a differential form $\omega \in \Omega^p(N)$ and a map $\varphi\colon M \longrightarrow N$, we have

$$\varphi^*(d\omega) = d(\varphi^*\omega).$$

closed differential form: a differential form for which the exterior derivative is zero.

exact differential form: a differential form that is the exterior derivative of a differential form.

B Definitions from Lie Group Theory

B.1 Lie Groups and Lie Algebras

Lie group: Let G be a group with law of composition $\cdot : G \times G \longrightarrow G$ given by $(g_1, g_2) \rightsquigarrow g_1 \cdot g_2 \equiv g_1 g_2$. A *real Lie group* G_L is a group that is also a manifold such that the mapping from any element to its inverse and the mapping from any two elements to their product are smooth maps. This last condition is equivalent to the condition that the mapping $(g_1, g_2) \rightsquigarrow g_1 g_1^{-1}$ is C^∞.

Group action: Let G be a group and let S be some set. Then G acts on S if there exists a map $G \times S \longrightarrow S$ defined by $(g, s) \rightsquigarrow s'$, where $g \in G$ and $s, s' \in S$ that satisfies the following axiom: $(gg', s) = (g, (g', s))$ for any $g, g' \in G$ and $s \in S$. In this case S is called a G-set.

Free action: Let S be a G-set. Then G acts freely on S if the only group element that satisfies the condition $gs = s$ for all $s \in S$ is the identity element of G.

right and left translations: The *right* and *left translations* of G_L are the diffeomorphisms of G_L labelled by the elements $g \in G_L$ and defined by:

$$r_g\colon\ G \longrightarrow G, \qquad g' \rightsquigarrow g'g \tag{16}$$

$$l_g\colon\ G \longrightarrow G, \qquad g' \rightsquigarrow gg'. \tag{17}$$

group representation: A *group representation* is a homomorphism $\rho\colon\ G \longrightarrow GL(V)$, where $GL(V)$ denotes the general linear group of V, viz. the space of linear transformations on V.

Lie group homomorphism: A homomorphism of Lie groups is a homomorphism of groups that is also smooth map between the underlying manifolds of the two Lie groups.

Lie group representation: A *Lie group representation* is a group representation where the homomorhpism ρ is a homomorphism of Lie groups.

Lie Algebra: If G is a Lie group, then the *Lie algebra* of G, denoted $\mathfrak{g}$, is the tangent space of the identity element of G, T_eG. T_eG is isomorphic to the space of all left-invariant vector fields $L(G)$, so both can be considered as $\mathfrak{g}$ the Lie Algebra of G.

left-invariant vector field: A vector field X on a Lie group G is *left-invariant* if it is l_g-related to itself for all $g \in G$, i.e.

$$l_{g*}X = X \text{ for all } g \in G,$$

where l_{g*} is the push forward map of l_g, defined by (17). Similarly a vector field X is *right-invariant* if it is r_g-related to itself for all $g \in G$. The set of all left-invariant vector fields on a Lie group G is denoted $L(G)$ and is a real vector space.

Lie bracket: for any two left-invariant vector fields X_1 and X_2, their *commutator* or *Lie bracket* $[X_1, X_2] \equiv X_1X_2 - X_2X_1$ is also a left-invariant vector field. The Lie bracket has a set of identifying properties, viz. antisymmetry, bilinearity and the Jacobi identity:

$$[A,[A',A'']] + [A',[A'',A]] + [A'',[A,A']] = 0, \qquad \text{for all } A, A', A'' \in \mathfrak{g}.$$

homomorphism of Lie Algebras: A homomorphism of Lie algebras is defined as a linear map $\rho\colon\ \mathfrak{g} \longrightarrow \mathfrak{h}$ such that $\rho([A,A']) = [\rho(A), \rho(A')]$ for all $A, A' \in \mathfrak{g}$.

C Definitions from the Theory of Fibre Bundles

C.1 Fibre Bundles

bundle: A *bundle* is defined to be a structure (E, π, M) consisting of a smooth manifold E, a smooth manifold M and an onto smooth map $\pi\colon E \longrightarrow M$.[28] The manifold E is called the *total space* or *bundle space*, the manifold M is called the *base space*, and π is called the *projection map*. For each point $p \in M$, the inverse image of p under π

$$E_p = \{q \in E : \pi(q) = p\}$$

is called the *fibre over* p.

fibre bundle: If for each $p \in M$ E_p is homeomorphic to a common space F, then F is known as the *fibre* of the bundle and the bundle is called a *fibre bundle*.

bundle morphism: A *morphism* of two bundles $\pi\colon E \longrightarrow M$ and $\pi'\colon E' \longrightarrow M'$ is a map $\psi\colon E \longrightarrow E'$ together with a map $\phi\colon M \longrightarrow M'$ such that ψ maps each fibre E_p into the fibre $E'_{\phi(p)}$. These maps have the property that $\pi' \circ \psi = \pi \circ \phi$.

bundle isomorphism: A bundle morphism is an isomorphism if both ϕ and ψ are diffeomorphisms.

restriction of a bundle: The *resriction* of a bundle $\pi\colon E \longrightarrow M$ to a submanifold $S \subseteq M$ is formed by total space $E|_S = \{q \in E | \pi(q) \in S\}$, base space S and projection map $\pi|_S$.

trivial bundle: bundles (E, π, M) that have the property that they are isomorphic to a *product bundle* $(M \times F, pr, M)$ for some space F.

locally trivial bundle: A bundle is called *locally trivial* with standard fibre F if for each point $p \in M$, there is a neighborhood U of p and a bundle isomorphism $\phi\colon E|_U \longrightarrow U \times F$.

section or cross-section: A section of a bundle (E, π, M) is a map $s\colon M \longrightarrow E$ such that the image of each point $p \in M$ lies in the fibre E_p over p.

vector bundle: An n-dimensional real vector bundle is a locally trivial fibre bundle (E, π, M) where each fibre E_p is an n-dimensional vector space provided that for each $p \in M$ there is a neighborhood U of p and a local trivialization $\phi\colon E|_U \longrightarrow U \times \mathbb{R}^n$ that maps each fibre E_p to the fibre $\{p\} \times \mathbb{R}^n$ linearly.

[28]The resriction that E, M and π be smooth technically makes the bundle in question a C^∞-*bundle*. For E and M topological manifolds and π continuous, (E, π, M) would be a *bundle*. But since we are will always have the former restrictions the will be no confusion caused by calling C^∞-bundles 'bundles.'

dual bundle: Given a vector bundle (E, π, M) one can construct the *dual bundle* (E^*, π^*, M), where

$$E^* = \bigcup_{p \in M} E_p^*,$$

where E_p^* is the dual space of E_p and $\pi^*\colon E* \longrightarrow M$ the map from each E_p^* to p.

direct sum vector bundle: given two vector bundles (E, π, M) and (E', π', M) it is possible to construct the *direct sum vector bundle* $E \oplus E'$ over M, where the fibre over $p \in M$ is $E_p \oplus E'_p$

tensor product vector bundle: given two vector bundles (E, π, M) and (E', π', M) it is possible to construct the *tensor product vector bundle* $E \otimes E'$ over M, where the fibre over $p \in M$ is $E_p \otimes E'_p$.

exterior algebra bundle: Given a vector bundle E one can define an *exterior algebra bundle* ΛE where the fibre over $p \in M$ is ΛE_p. The sections of the exterior algebra bundle form an algebra when fitted with a wedge product.

G-bundle: See Appendix D.

T **lives in** G**:** if $p \in U_\alpha$, then T *lives in* G if it is of the form

$$[p, v]_\alpha \longmapsto [p, gv]_\alpha$$

for some $g \in G$.

T **lives in** $\mathfrak{g}$**:** Consider the Lie algebra $\mathfrak{g}$ of G, $T\colon E_p \longrightarrow E_p$ *lives in* $\mathfrak{g}$ if it is of the form

$$[p, v]_\alpha \longmapsto [p, d\rho(x)v]_\alpha$$

for some $x \in \mathfrak{g}$.

endomorphism: Given a vector space V the linear functions from V to itself are called *endomorphisms*. The vector space of all endomorphisms is denoted $\mathrm{End}(V)$, following [2].

endomorphism bundle Given a vector bundle E over a manifold M, the endomorphism bundle $\mathrm{End}(E)$ is the bundle $E \otimes E^*$.

gauge transformation: if (E, π, M) is a G-bundle, where G is some Lie group, and $T \in \mathrm{End}(E)$ then T is a *gauge transformation* if $T(p)$ lives in G for all $p \in M$. The set of all gauge transformations forms a group, which we will denote $\mathcal{G}$, to distinguish it from the group G of the G-bundle.

C.2 Connection and Curvature

connection: Suppose that (E, π, M) is a vector bundle and v is a vector field, then a *connection* on M is a function D_v: $\Gamma(E) \longrightarrow \Gamma(E)$, $s \rightsquigarrow D_v s$, that is C^∞-linear in v and real-linear[29] in $s \in \Gamma(E)$ and satisfies the Leibniz law

$$D_v(fs) = v(f)s + fD_v s,$$

for any $v \in \mathbb{V}(M), s \in \Gamma(E)$ and $f \in C^\infty(M)$.

covariant derivative: $D_v s$ is called the *covariant derivative* of s in the direction of v.

vector potential: the vector potential A is

$$A = A^i_{\mu j} e_i \otimes e^j \otimes dx^\mu,$$

and by suppressing the 'internal indicies' i and j the components can be written as

$$A_\mu = A^i_{\mu j} e_i \otimes e^j.$$

standard flat connection: any connection D can be written as $D^0 + A$:

$$D_v s = (v(s^i) + A^i_{\mu j} v^\mu s^j) e_i = D^0_v s + A(v)s.$$

The connection D^0 is called the *standard flat connection*.

parallel transport: Suppose (E, π, M) is a vector bundle with a connection D defined on it. Let γ: $[0, T] \longrightarrow M$ be a smooth map from p to q and suppose that for $t \in [0, T]$, $u(t)$ is a vector in the fibre of E over $\gamma(t)$. Then $u(t)$ is *parallel transported* along γ if the following condition holds:

$$D_{\gamma'(t)} u(t) = \frac{d}{dt} u(t) + A(\gamma'(t)) u(t).$$

covariant derivative The object $D_{\gamma'(t)} u(t)$ in the previous definitoin is called the *covariant derivative.*

curvature Given a vector bundle (E, π, M), two vector fields v and w on M and a conneciton D, the curvature $F(v, w)$ is defined to be the operator on sections of E defined by

$$F(v, w)s = D_v D_w s - D_w D_v s - D_{[v,w]} s,$$

where the last term is included to correct for the fact that the covariant derivatives may fail to commute because the vector fields v and w fail to commute, i.e. have non-vanishing Lie bracket. The curvature is C^∞-linear over v, w and s.

[29] Or complex-linear if E is a complex vector bundle

flat connection A connection with curvature zero, $F(v, w) = 0$ for any v, w and s, is a *flat conneciton.*

D Construction of G-bundles

Let $\{U_\alpha\}$ be an open cover of M, let V be a vector space and let ρ be a representation of a group G on V. It is possible to create a fibre bundle structure (E, π, M) by 'gluing' together all the trivial bundles $U_\alpha \times V$ using *transition functions* $g_{\alpha\beta}$: $U_\alpha \cap U_\beta \longrightarrow G$. The bundle space is formed by taking the disjoint union

$$\bigcup_\alpha U_\alpha \times V,$$

and identifying points $(p, v) \in U_\alpha \times V$ and $(p, v') \in U_\beta \times V$ if

$$v = \rho(g_{\alpha\beta}(p))v', \tag{18}$$

which can be written as

$$v = g_{\alpha\beta}v' \tag{19}$$

as a useful shorthand. The conditions $g_{\alpha\alpha}$ on U_α and $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1$ on $U_\alpha \cap U_\beta \cap U_\gamma$ are required for the consistency of this operation. If we let $[p, v]_\alpha$ denote the point of E corresponding to $(p, v) \in U_\alpha \times V$, then the projection map is defined by $\pi[p, v]_\alpha = p$. It can be shown that the resulting structure is a vector bundle. Such a bundle is called a *G-bundle* with *gauge group* G and V the *standard fibre.*